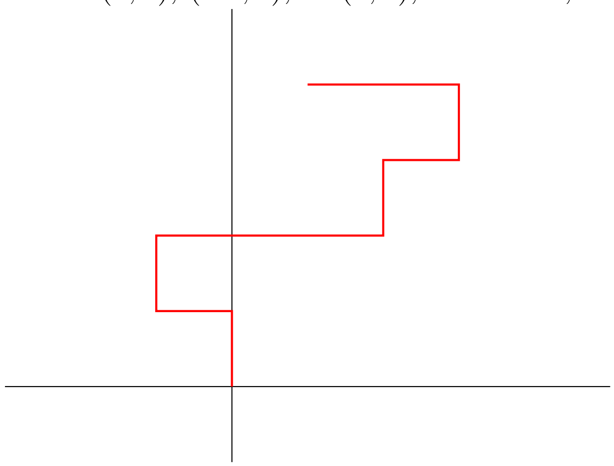
Example 0.1 (Enumerative Counting Problems)**.** Let $f(n)$ be the number of n-step paths starting at the origin and taking non-intersecting steps in the directions $(0,1)$, $(-1,0)$, or $(1,0)$, labeled N, W, and E respectively.



in 11 steps

Then we can ditch the picture and instead count strings of $\{N, W, E\}$-letters where EW or WE does not appear as a factor.

Let's try to express $f(n)$ in terms of $f(k)$ for $k < n$, trying for polynomial coefficients. We can start from the end. A word of length n ends in:

$$\begin{cases} N & f(n-1) \text{ of these} \\ NW & f(n-2) \text{ of these} \\ NE & \\ WW & f(n-1) \text{ of these } NE, WW, \text{ and } EE \text{ combined} \\ EE & \end{cases}$$

Then $\star N$ becomes $\star NE$, $\star W$ becomes $\star WW$, and $\star E$ becomes EE when at the end of a word to give the three NE, WW, EE cases becoming $f(n-1)$.

Alternatively, we can brute force it to get the first fer tewms to find a recurrence with linear algebra. Then $f(n) = 2f(n-1) + f(n-2)$ where $f(0) = 1$ and $f(1) = 3$. Then we have

$$\sum_{n\geq 2} f(n)x^n = 2\sum_{n\geq 2} f(n-1)x^n + \sum_{n\geq 2} f(n-2)x^n.$$

Note that we can put $G(x) = \sum_{n\geq 0} f(n)x^n = 1 + 3x + \ldots$, so

$$G(x) - 1 - 3x = \sum_{n\geq 2} f(n)x^n, 2x[G(x) - 1] = 2\sum_{n\geq 2} f(n-1)x^n, \text{ and } x^2G(x) = \sum_{n\geq 2} f(n-2)x^n$$

so we can rewrite our equation as

$$G(x) - 1 - 3x = 2x[G(x) - 1] + x^2 G(x).$$

Then we can do some algebra to get

$$G(x) = \frac{1+x}{1-2x-x^2} = \frac{1+x}{(1-(1+\sqrt{2})x)(1-(1-\sqrt{2})x)} = \frac{A}{1-(1+\sqrt{2})x} + \frac{B}{1-(1-\sqrt{2})x}$$

Next clear the denominators and compare coefficients and expand these geometric series to get

$$G(x) = \sum_{n \geq 0} \left[\frac{1}{2}(1+\sqrt{2})^{n+1} + \frac{1}{2}(1-\sqrt{2})^{n+1} \right] x^2.$$

Note that $1 + \sqrt{2} \approx 2.414$ and $1 - \sqrt{2} \approx -0.414 < 1$, so $f(n) = \frac{1}{2}(1 + \sqrt{2})^{n+1} + o(1)$.

In general, growth of the terms $\{a_n\}$ depends on the closest pole of $G(x) = \sum a_n x^n$ to the origin, if $G(x)$ has isolated singularities.

Remark: $G(x) = \frac{p(x)}{q(x)} = \sum f(n) x^n$. Then there should be a linear recurrence for $f(n)$. If q has distinct roots, then recurrence has constant coefficients. There should be an explicit formula for $f(n)$ in terms of the roots of $q(x)$.