## 1 Math 607: Day 10

### 1.1 More Branching Processes

Recall that $\phi(u)=\mathbb{E}\left[u^{x}\right]=\sum_{n=0}^{\infty} p_{n} u^{n}$. If we remind ourselves that: $\phi(0)=p_{0}, \phi(1)=1$, and $\phi^{\prime}(1)=\mu$, then we see that $\phi(u)$ has a fixed point on the interval $[0,1]$ whenever $\mu>1$. This results in the following theorem.

Theorem 1. If $\mu>1$, then there exists a fixed point of $\phi$ at $1-q$ for some $q \in[0,1]$. In this case, $q$ is the probability that the branching process survives. Conversely, if $\mu \leq 1$ then the branching process dies.

We omit the proof of the theorem and instead provide an example.
Example 1. Let $X \sim \operatorname{Poisson}(\lambda)$ so $P(X=n)=\frac{e^{-\lambda} \lambda^{n}}{n!}$. Then

$$
\begin{aligned}
\phi(u) & =\sum_{n \geq 0} \frac{u^{n} e^{-\lambda} \lambda^{n}}{n!} \\
& =e^{-\lambda} e^{u \lambda} \\
& =\exp (\lambda(u-1))
\end{aligned}
$$

Note that $\mathbb{E}[X]=\phi^{\prime}(1)=\lambda$ so the branching process dies off if $X$ has a rate less than or equal to one.
We now develop some general theory for approximating $q$. Suppose that $\mu=1+s$ for some small s. Let $m=\mathbb{E}[X(X-1)]$. Expanding $\phi(u)$ about 1 gives us

$$
\begin{aligned}
& \phi(1-x)=\phi(1)-x \phi^{\prime}(1)+\frac{x^{2}}{2} \phi^{\prime \prime}(1)+O\left(x^{3}\right) \\
& \cong 1-\mu x+\frac{x^{2}}{2} m+O\left(x^{3}\right)
\end{aligned}
$$

So $1-x=\phi(1-x)$ is solved whenever $x *=\frac{2 s}{m}$. Let's use this approximation in an example!
Example 2. Each virus particle that enters your mouth may either die or reproduce with the probability of surviving being $p=.01$. Every reproducing virus produces 150 copies of itself. What is the chance that you get sick?

If we assume that you get sick whenever the branching process "survives," then we can leverage our previous work to solve this problem. First we compute $\mu$. As a virus has a survival probability of .01 and produces 150 offspring, $\mu=150 * .01=1.5$. As $\mu>1$, this branching process has a shot at surviving.

To find $q$ we employ our approximation. Quickly we see that $s=.5$. Calculating $m$ is fairly straightforward

$$
m=\mathbb{E}[X(X-1)] \cong 150^{2} * .01
$$

So $q \cong \frac{2 * 5}{.01 * 150^{2}} \cong .01$. So you have a roughly one percent chance of getting sick.

### 1.2 Random Graphs

We now pivot to discuss random graphs. First a definition!
Definition 1. For an undirected graph $G=(V, E)$ with $N$ nodes let $d_{n}=\frac{\# \text { of nodes with degree n }}{N}$ and $D=$ $\operatorname{deg}$ (randomly selected node). In other words, $P(D=n)=d_{n}$. We call $d_{n}$ the degree sequence of $G$.

Given a sequence of graphs $\left(G_{n}\right)$ the sequence is sparse if $D^{N} \rightarrow D$ in distribution as $N \rightarrow \infty$. In other words, if the degree sequences converge to some nice probability distribution.

Example 3. $\mathbb{Z}^{2} \cap[0, \sqrt{n}]$ defines a nice sparse sequence of graphs.
Example 4 (Non-Example). Connect $u, v \in\{1, \ldots, N\}$ if $u \mid v$ or $v \mid u$. As $N \rightarrow \infty, d_{n} \rightarrow \infty$.
Here is a theorem about random graphs. It is colloquially known as the "Your Friends Have More Friends Than You" theorem.
Theorem 2. Assume that $d_{0}=0$. Let $u$ be a uniform picked node and $v$ a uniformly selected neighbor of $u$. Then $\mathbb{E}[\operatorname{deg}(v)] \geq \mathbb{E}[\operatorname{deg}(u)]$.

Proof. Note that $P($ pick edge $(\mathrm{u}, \mathrm{v}))=\frac{1}{N} \frac{1}{\operatorname{deg}(u)}$ for all $(u, v) \in E$. Now

$$
\mathbb{E}[\operatorname{deg}(u)]=\sum_{u, v} \frac{\operatorname{deg}(\mathrm{u})}{N} \frac{1}{\operatorname{deg}(\mathrm{u})}=\frac{\# \text { of edges }}{N}
$$

Likewise

$$
\mathbb{E}\left[\operatorname{deg}(v)=\sum_{u, v} \frac{\operatorname{deg}(v)}{N} \frac{1}{\operatorname{deg}(u)}\right.
$$

So

$$
\mathbb{E}[\operatorname{deg}(u)] \leq \sum_{u, v} \frac{1}{2 N}\left(\frac{\operatorname{deg}(u)}{\operatorname{deg}(v)}+\frac{\operatorname{deg}(v)}{\operatorname{deg}(u)}\right) \leq \mathbb{E}[\operatorname{deg}(v)]
$$

