# MATH 607 (Applied Math I) Lecture Notes

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October 23, 2019

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Today we're going to talk more about branching processes. For setup, we have a collection of independent, identically distributed random variables, which we generically denote $X$. Each random variable is the number of offspring that each parent has. The generating function $\phi(u)$ is given by

$$\phi(u) = \mathbb{E}[u^X] = \sum_{n\geq 0} u^n p_n,$$

where $p_n = \mathbb{P}\{X = n\}$ is the probability that a parent has $n$ children. (As the $X$ are identically distributed, this probability does not depend on the parent.) We can think of this generating function as a map $\phi : [0,1] \to [0,1]$. For example, $\phi(0) = p_0$, while

$$\phi'(1) = \mu = \mathbb{E}[X] = \sum_n np_n,$$

where $\mu$ is the mean (or expectation) of the distribution used for the family $X$.

**Theorem.** *If $\mu > 1$, then there is a fixed point of $\phi$, perhaps $1-q$ for some $0 < q \leq 1$. That is, $\phi(1-q) = 1-q$. Furthermore, $q$ is the survival probability of the branching process,*

$$q = \mathbb{P}\{\textit{Survival of BP}\} = \mathbb{P}\left\{\lim_{t\to\infty} N_t > 0\right\}.$$

*Conversely, if $\mu \leq 1$, then*

$$\mathbb{P}\{N_t \to 0\} = 1.$$

*Proof.* We define

$$1 - q_t = \mathbb{P}\left\{N_t > 0\right\} = \phi^{(t)}(0) = \phi(\phi(\ldots\phi(0)\ldots)).$$

We note that $\phi(u) \nearrow$ as $u \nearrow$. So if $u < 1-q$ then $u < \phi(u) < 1-q$. $\square$

**Example.** *Suppose that $X$ has the Poisson distribution with mean $\lambda$. This means that*

$$\mathbb{P}\{X = n\} = e^{-\lambda}\frac{\lambda^n}{n!}.$$

*Then*
$$\phi(u) = \sum_{n\geq 0} u^n e^{-\lambda}\frac{\lambda^n}{n!} = e^{-\lambda}e^{-\lambda u} = \exp(\lambda(u-1)),$$
*and we note that* $\mathbb{E}[X] = \phi'(1) = \lambda$*, which is why* $\lambda$ *is referred to as the mean of this distribution.*

**Approximating the Survival Probability.** Suppose that $\mu = 1+s$ for some small-ish $s$. Then we can build a Taylor series around $x=0$ for $\phi(1-x)$;

$$\phi(1-x) = \phi(1) - x\phi'(1) + \frac{x^2}{2}\phi''(1) - \dots.$$

here we see that $\phi''(1) = \mathbb{E}[X(X-1)]$ is almost the variance. We call this quantity $m$. Using the quadratic approximation of this Taylor series, we see that

$$\phi(1-x) \approx 1 - \mu x + \frac{m}{2}x^2.$$

Since $1-q$ is a fixed point of $\phi$, $q$ will (approximately) be a solution to the quatratic equation

$$1 - x = 1 - \mu x + \frac{m}{2}x^2.$$

Since this equation is solved by $x = 2s/m$, we see that $q \approx 2s/m$ is a nice approximation for the survival probability when $s$ is small. When $s$ is larger, this Taylor expansion doesn't converge very fast, so the quadratic approximation won't be the best.

**Example.** *Each virus particle that goes in your mouth might die or reproduce. Suppose that* $\mathbb{P}\{\textit{death}\} = 99\%$*, and*
$$\mathbb{E}[\ \#\ \textit{offspring} \mid \textit{not death}\ ] = 150.$$
*Question: What is the probability, approximately, that you get sick from a single particle, i.e., that the number of viruses* $\nearrow \infty$*?*

*We note that* $\mu = 150\cdot 0.1 = 1.5 > 1$*, so* $s = 0.5$*. To calculate* $m$*, we use* $m = \mathbb{E}[X(X-1)]$*. This is 0 with 99% probability, and* $\approx 150^2$ *with probability 1%. So we see that*

$$q \approx \frac{2s}{m} \approx \frac{2(0.5)}{0.01(150^2)} \approx 1\%.$$

**Random Graphs.** For an undirected graph with $N$ nodes, let $d_n^{(N)}$, be the proportion of nodes with degree $n$, and let $D^{(N)}$ be a random variable giving the degree of a uniformly randomly chosen node in the graph. $d_n^{(N)}$ is called the degree sequence. We note that $\mathbb{P}\{D^{(N)} = n\} = d_n^{(N)}$.

**Definition.** *A sequence of graphs is called* sparse *if* $D^{(N)}$ *converges in distribution as* $N \to \infty$*, perhaps to a random variable* $D$*. In this case, convergence in distribution means that* $d_n^{(N)}$ *converges for each* $n$ *as* $N \to \infty$ *to a possible degree sequence, that is, it converges to some* $d_n$ *with* $\sum d_n = 1$*.*

**Example.** *A branching provess with $\mu > 1$, run until it has $N$ nodes.*

**Example.** $\mathbb{Z}^2 \cap [0, \sqrt{n}]^2$. *Then as $N \to \infty$, $d_i^{(N)} \to 0$ for $i \neq 4$ while $d_4^{(N)} \to 1$.*

**Non-Example.** *Connect $u, v \in \{1, 2, \ldots, n\}$ if $u|v$ or $v|u$. This is not sparse as $\mathbb{P}\{D^{(n)} > x\} \to 1$ as $N \to \infty$ for all $x$.*

**Theorem** (”Your friends have more friends than you do.”)**.** *Assume that $d_0 = 0$. Let $U$ be a uniformly chosen node and $V$ a uniformly chosen neighbor of $U$. Then*

$$\mathbb{E}[\deg(V)] > \mathbb{E}[\deg(U)].$$

*Proof.* The probability that the edge $(U, V)$ is chosen is given by

$$\mathbb{P}\{ \text{ pick edge } (U, V) \ \} = \frac{1}{N} \frac{1}{\deg(U)}.$$

for any edge $(U, V)$. We note that

$$\mathbb{E}[\deg(U)] = \sum_{(U,V) \in E} \deg(U) \frac{1}{N} \frac{1}{\deg(U)} = \frac{2|E|}{N},$$

which is the average degree. Furthermore,

$$\mathbb{E}[\deg(V)] = \sum_{(U,V) \in E} \deg(V) \frac{1}{N} \frac{1}{\deg(U)}.$$

To evaluate this, we note that one has the fundamental inequality

$$\frac{1}{2} \left( \frac{x}{y} + \frac{y}{x} \right) \geq 1,$$

which holds for every $x, y > 0$. So

$$\mathbb{E}[\deg(U)] = \sum_{U,V} \frac{1}{N} \leq \sum_{U,V} \frac{1}{N} \left( \frac{\deg(U)}{\deg(V)} + \frac{\deg(U)}{\deg(V)} \right) \frac{1}{2} = \mathbb{E}[\deg(V)].$$

□