Applied Day 12

October 28, 2019

Peter Ralph lecturing.

1 Configuration model

A random graph such that every degree $\sim D$ with probability given by $\mathbb{P}\{D = n\} = d_n$. Let

$$\mu = \frac{\mathbb{E}[D(D-1)]}{\mathbb{E}[D]}$$
$$= \sum_{n \geq 0} \frac{n d_n}{\sum_{m \geq 0} m d_m}(n-1)$$

be the mean of the the "size-biased distribution of $D-1$", ie picking a vertex with probability proportional to the number of edges connected to it. We consider the generating function of D and the sized-biased version of D:

$$\Phi(z) = \mathbb{E}[z^D] = \sum_{n \geq 0} z^n d_n$$

$$\Psi(z) = \frac{\mathbb{E}[z^{D-1} D]}{\mathbb{E}[D]} = \sum_{n \geq 0} z^n \frac{n d_n}{\sum_{m \geq 0} m d_m}$$

Recall from homework 1 that $\mu = \Psi'(1)$.

At step t, let

$$A_t = \text{"active vertices"} \quad (\text{where } A_0 = 0)$$
$$V_t = \text{"visited vertices"} \quad (\text{note } |V_t| = t)$$
$$U_t = \text{"unvisited vertices"} \quad [N] \setminus (A_t \cup V_t)$$

We explore by

1. taking an active vertex
2. move it to V
3. adding its children[1] to A

[1] Children isn't quite the right word... we want to add the vertices it's connected to which aren't already in A or V

We will compare this to a branching process.

Let $N_0 = 1$. Want: N_t to give us the size of a branching process so far.

Note: branching process goes generation by generation, so we defined N_t slightly different.

Let $N_t = |A_t| + E_t + t$ where E_t is the "excess". This is not quite a branching process since we get some cross connections, e.g. a node having muliple parents. We want it to look as much as possible like an exploration process; when we hit a trouble edge we do (something).

$$E_0 = 0$$

$$\begin{aligned} E_{t+1} &= \#(\text{edges between vertices in } V_t \text{ except the original ones}) \\ &= \#(\text{edges between vertices in } V) - (t-1) \end{aligned}$$

Note: # edges in a tree with vertices is $n - 1$.

<u>Claim:</u> Let $(X_k)_{k\geq 0}$ be a branching process with $\mathbb{P}\{X_0 = n\} = d_n$ and offspring distribution with generating function $\Psi(z)$. Then

$$\left(\lim_{t\longrightarrow\infty} N_t\right) = N_\infty$$

has the same distribution as

$$\left(\lim_{k\longrightarrow\infty} \sum_{s=0} X_s = T_k\right) = T_\infty$$

the total size of the branching process.

Proof. This is obvious (???) □

Proof.

1. Let's number each vertex v with a "generation" $g(v)$ so that

$$g(v) = g(\text{parent of } v) + 1$$

   i.e. $g(v)$ is the min of $g(\text{parents of v}) + 1$

2. Let $X_k = \#\{\text{vertices } v : g(v) = k\} + (\text{something else related to } E)$ where

$$(\text{something})_k = \sum_{l=0}^{k} Y_{l,(k-l)}$$

   the number of edges from generation k to generation k.
   Let $B_k = \#\{\text{edges in excess connecting to generation k}\}$

   For $t \geq 0$, $0 \leq j \leq B_t - 1$, let Y_{tj} be an independent branching process with the same distribution as X. Then $(\text{something})_k$ counts how many extra things were added in this new branching process.

Then the idea of this proof is that in the limit the amount of excess should go to zero (?).

$N_\infty \geq |A_\infty + T$ where $A_\infty = 0$, the number of active vertices at the end of the exploration process, and T is the size of component and $1 = |C|$ so if N_∞ is finite then so is $|C|$. We assume the connected component is finite. $\square$

Fact: Proportion of nodes not in the giant component is asymptotically the probability of extinction of the branching process.