Applied Notes

November 6th, 2019

Example: Let K_n be the complete graph on n vertices. Then the number of spanning trees of K_n is $\det \Delta_i^i$ for any i. If we choose $i = n$, we have

$$\det \Delta_i^i = \det \begin{bmatrix} n-1 & -1 & \cdots & -1 \\ -1 & n-1 & \cdots & -1 \\ \vdots & \vdots & \ddots & \vdots \\ -1 & -1 & \cdots & n-1 \end{bmatrix}.$$

Subtracting row 1 from each other row,

$$\det \Delta_i^i = \det \begin{bmatrix} n-1 & -1 & -1 & \cdots & -1 \\ -n & n & 0 & \cdots & 0 \\ -n & 0 & n & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -n & 0 & 0 & \cdots & n \end{bmatrix}.$$

Now adding every column except column 1 to column 1,

$$\det \Delta_i^i = \det \begin{bmatrix} 1 & -1 & -1 & \cdots & -1 \\ 0 & n & 0 & \cdots & 0 \\ 0 & 0 & n & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & n \end{bmatrix}.$$

By a theorem from Cauchy, this determinant is equal to n^{n-2}, so there are n^{n-2} spanning trees of K_n.

Definition 0.1: Let G be an undirected graph and let $e = (e_0, e_1)$ and $f = (f_0, f_1)$ be arbitrarily oriented edges in G. Let $\overleftarrow{f} = (f_1, f_0)$. $J^e(f) = \mathbb{E}(\#$ times f is used in a random walk from e_0 to $e_1) = \mathbb{E}\left(\# \text{ times } \overleftarrow{f} \text{ is used}\right)$. $\beta(e, f) = \mathbb{P}($the path from e_0 to e_1 in a uniformly random spanning tree of G uses $f)$.

Theorem 0.2: $\beta(e, f) - \beta\left(e, \overleftarrow{f}\right) = J^e(f)$.

Proof: $\beta(e,f) - \beta\left(e, \overleftarrow{f}\right)$ is the expected number of times a loop-erased random walk from e_0 from e_1 uses f, minus the amount of times it uses $\overleftarrow{f}$. This is because a loop-erased random walk either uses f once or not at all, so the probability is the same as the expectation. But in the non-loop-erased walk, the probability of walking through f forwards is the same as the probability of walking through it backwards, so the expected number of times f is used inside of a loop is the same as the expected number of times that $\overleftarrow{f}$ is. Thus $\beta(e,f) - \beta\left(e, \overleftarrow{f}\right)$ is the expected number of times f is used outside of a loop in a random walk from e_0 to e_1, minus the expected number of times $\overleftarrow{f}$ is used. But this is exactly $J^e(f)$.

Comment: J^e describes the current flowing through a graph G when a 1V battery is placed on edge e and 1Ω resistors are placed on every edge.