Combinatorics and Computation Notes

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1 Example: Aztec Diamond Graph

We begin with an example of the Kasteleyn matrix using a bipartite Aztec Diamond graph of order 2, as shown in Figure 1. Let $G = (V, E)$ be this graph and let the sign function $\sigma : E \to \{\pm 1\}$ be defined by the figure with unlabelled edges mapping to $+1$. Note that this map satisfies the defining property that

$$\prod_{i=1}^{2k} \sigma(e_i) = \begin{cases} 1 & k \text{ odd} \\ -1 & k \text{ even} \end{cases}$$

for each face $F = \{e_1, \ldots, e_{2k}\}$ defined by G.

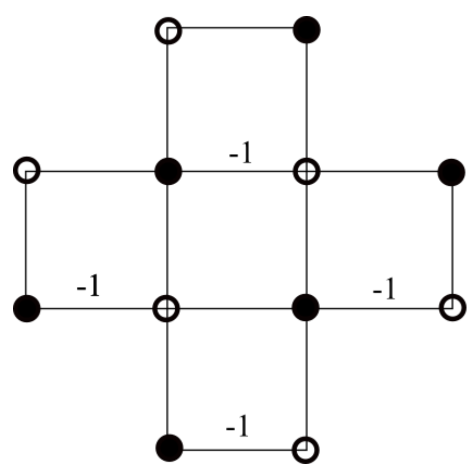

Figure 1: A Bipartite Aztec Diamond of Order 2

We can also consider the order n Aztec Diamond graph. Figure 2 shows the case $n = 3$. In general, the order n Aztec Diamond graph has

$$2^{\binom{n+1}{2}} \qquad \text{perfect matchings.}$$

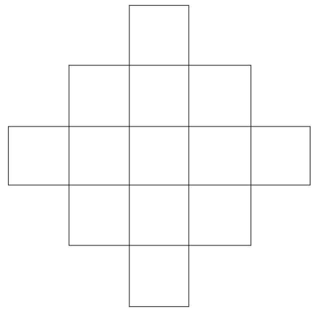

Figure 2: Aztec Diamond of Order 3

2 Edge - Placement Probability

We now study the probability that a given edge in a bipartite graph appears in a random perfect matching. Specifically, let $G = (V = V_\bullet \amalg V_\circ, E)$ be an embedded, planar, bipartite graph. We assume that G has perfect matchings, so in particular $|V_\bullet| = |V_\circ|$.

Let K be its Kasteleyn matrix so that K is an $|V_\bullet| \times |V_\circ|$ matrix whose rows are indexed by black vertices and columns by white vertices. Then

$$|\det K| = \#\{\text{perfect matchings on } G\}.$$

Suppose $e \in E$. We would like to know how to compute $\mathbb{P}\{e \in M\}$ where M is a uniformly random perfect matching on G.

2.1 Motivation

Let us see how this probability arises in studying Aztec Diamonds. Inspection shows that for large n, uniformly random perfect matchings on the Aztec Diamond tend to follow a particular patter as shown in Figure 3. Within the circle, the matching M appears random. However, outside of the circle, we find a remarkably regular "frozen grid" pattern.

It can be shown that as $n \to \infty$ the probability of this pattern occurring goes to 1. To show this, we compute $\mathbb{P}\{e \in M\}$ for a fixed horizontal edge e near the boundary of the Aztec Diamond. We eventually find that $\mathbb{P}\{e \in M\} \to 0$ as $n \to \infty$, thus leaving only vertical edges.

2.2 Computation

Let us compute the desired probability. Set G, E, V, and M as before. Let $e \in E$ be arbitrary. We have

$$\mathbb{P}\{e \in M\} = \frac{\#\{M \mid e \in M\}}{\#\{M\}} = \frac{\#\{M \mid e \in M\}}{|\det K|}.$$

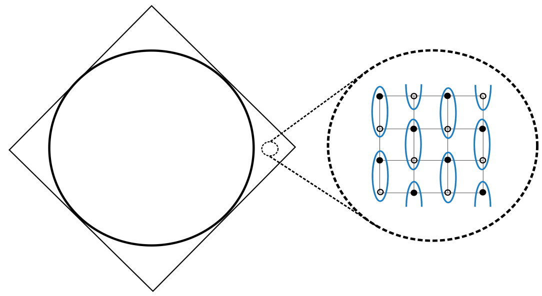

Figure 3: Aztec Diamond Limit Patter

We must determine $\#\{M \mid e \in M\}$.

Define $G' = (V', E')$ where

$$V' = V \setminus \{\text{endpoints of } e\} \quad \text{and} \quad E' = E \setminus (\{e\} \cup \{\text{all edges incident to } e\}) \,.$$

Let M' represent a perfect matching on G'. Then

$$\#\{M \mid e \in M\} = \#\{M'\}.$$

The equality is easily observed by drawing a minimal example.

We note that, as G is bipartite, the vertices removed from V are of opposite colors. Removing them thus corresponds to removing one column and one row from K. It can be shown that the resulting matrix K' is indeed the Kasteleyn matrix for G'.

With this in mind, suppose that e has endpoints $(a, b) \in V_\bullet \times V_\circ$. Then $K' = K_a^b$, denoting K with row a and column b removed. We thus have

$$\#\{M'\} = |\det K_a^b|$$

whence

$$\mathbb{P}\{e \in M\} = \frac{|\det(K_a^b)|}{|\det(K)|} = |(K^{-1})_{ba}|$$

where $(K^{-1})_{ba}$ denotes the bath entry of K inverse. The right-most equality follows from Cramer's Rule. In words, the probability that a given edge e in a bipartite, planar graph G, with endpoints $a \in V_\bullet$ and $b \in V_\circ$ is equal to the bath entry of K^{-1}.

3 Dimers and Spanning Trees (Temperley Map)

We will now observe a connection between perfect matchings and spanning trees. Specifically, we will show that spanning trees of a planar graph G are in bijection with the perfect matchings on the double graph of G.

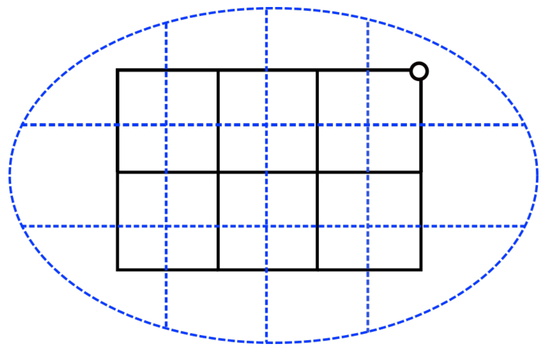

Figure 4: Grid Graph and its Dual

Consider a planar, bipartite graph $G = (V, E)$ and its dual $G^* = (V^*, E^*)$. Distinguish a vertex $v_0 \in V$ and let $v_\infty \in E^*$ represent the "outside" vertex from G. Considering G as a compact subspace of $\mathbb{R}^2$, we can consider v_∞ to be the point ∞ in the one point compactification $\mathbb{R}^2 \cup \{\infty\}$.

The case that G is a 2×3 grid is shown in the Figure 4. There, G is drawn with solid lines and G^* is drawn with dotted lines. The distinguished vertex v_0 is shown with an open circle. The vertex v_∞ is represented by the outermost oval.

Pick a spanning tree T for G and let T^* be the dual spanning tree for G^*. That T^* is dual to T means that the edges of T and T^* do not intersect. It is straightforward to check that T^* always exists. Figure 5 shows T in yellow and T^* in green.

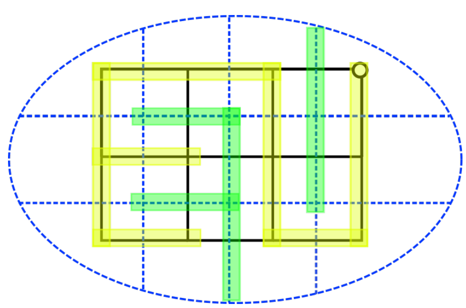

Figure 5: Dual Spanning Trees

We now define the **double graph of** G to be the graph $G_d = (V_d, E_d)$ with

$$V_d = (V \amalg V^* \amalg \{\text{all intersections of an edge } e \text{ with its dual } e^*\}) \setminus \{v_0, v_\infty\}.$$

The edge set E_d is obtained as a subdivision of $E \amalg E^*$ according to the intersections of each edge with its dual. This process is shown in Figure 6, so that each pair (e, e^*) contributes four edges to E_d. Note that the definition of G_d

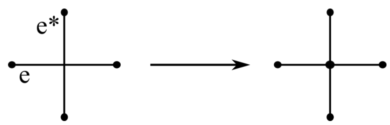

Figure 6: Subdivision Process for Double Graph

depends upon the vertex v_0.

We can now define the correspondence between the spanning trees on G and the perfect matchings on G_d. Let T and T^* be as before and identify them with their images in G_d. Starting at the leaves of T and T^*, match pairs of vertices in G_d along each tree until all vertices are matched. The result will be a perfect matching on G_d.

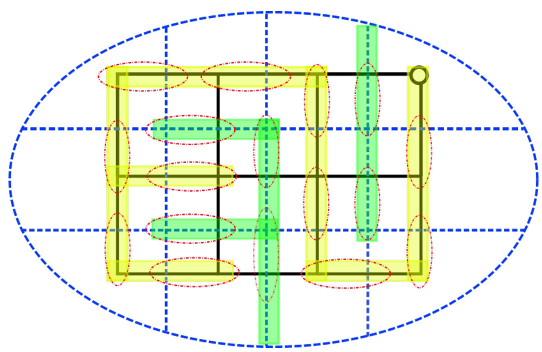

Figure 7: Subdivision Process for Double Graph

This process defines a bijection between the spanning trees of G and the perfect matchings of G_d. This bijection is due to Temperley. Figure 7 shows the perfect matching on G_d from our example.