Applied Notes

November 20th, 2019

Proof: (continued) We showed previously that the limit $\lim_{T\to\infty} \hat{\pi}_i^{(T)}$ exists. Now let $u_i^{(T)}$ be the expected number of visits to i with $1 \le t \le T$. Since $|\{t \mid 1 \le t \le T, X(t) = i\}| = \sum_{t=1}^{T} \mathbb{1}_{X(t)=i}$, where

$$\mathbb{1}_{X(t)=i} = \begin{cases} 1, & X(t) = i \\ 0, & \text{otherwise} \end{cases},$$

$u_i = \mathbb{E}\left[\sum_{t=1}^{T} \mathbb{1}_{X(t)=i}\right] = \sum_{t=1}^{T} \mathbb{E}\left[\mathbb{1}_{X(t)=i}\right] = \sum_{t=1}^{T} \mathbb{P}(X(t) = i)$. Now $\mathbb{P}(X(t) = i) = \sum_j \mathbb{P}(X(t-1) = j)P_{ji}$, so

$$u_i = \left(\sum_j \sum_{t=1}^{T-1} \mathbb{P}(X(t) = j)P_{ji}\right) + \mathbb{P}(X(1) = i).$$

But $\sum_{t=1}^{T-1} \mathbb{P}(X(t) = j) = u_j - \mathbb{P}(X(T) = j)$, so

$$u_i = \left(\sum_j u_j P_{ji}\right) - \left(\sum_j \mathbb{P}(X(T) = j)P_{ji}\right) + \mathbb{P}(X(1) = i).$$

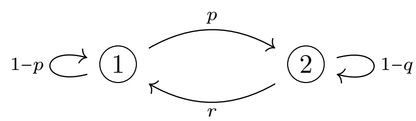
Since $\mathbb{E}\left[\hat{\pi}^{(T)}\right] = \frac{1}{T}u^{(T)}$, $\mathbb{E}\left[\hat{\pi}_i{}^{(T)}\right] = \sum_j P_{ji}\mathbb{E}\left[\hat{\pi}_j{}^{(T)}\right] + O\left(\frac{1}{T}\right)$. Thus as $T \longrightarrow \infty$, $\lim_{T\to\infty} \hat{\pi}_i^{(T)} = \hat{\pi}_i$.

Definition 1: A Markov chain X with transition matrix P is **ergodic** if for all pairs (i, j), $P_{ij}^n > 0$ for some n.

Definition 2: A Markov chain X with transition matrix P is **acyclic** if $\gcd\{n \in \mathbb{N} \mid P^n{}_{ii} > 0\} \neq 1$.

Proposition 3: If a transition matrix P is ergodic and acyclic, then there is a unique solution to $\pi P = \pi$ that satisfies $\sum_i \pi_i = 1$ and $\pi_i > 0$.

Example: Consider a Markov chain with only two states:



Then the transition matrix is $P = \begin{bmatrix} 1-p & p \\ q & 1-q \end{bmatrix}$, and a stationary distribution is $\pi = \begin{bmatrix} \frac{q}{p+q} & \frac{p}{p+q} \end{bmatrix}$.

Theorem 4: (Perron-Frobenius) Every transition matrix P has eigenvalues with magnitude no greater than 1, and P has at least one left nonnegative left eigenvector with eigenvalue 1.

Example: Move a king randomly on a chessboard, picking uniformly between legal moves at each time step. What proportion of time will it spend in the four corners?

The Markov chain is clearly reversible, so if we let l_i be the number of legal moves out of square i, then if $i \longrightarrow j$ is a legal move, $P_{ij} = \frac{1}{l_i}$. Thus

$$\pi_i = \frac{l_i}{\sum_j l_j}.$$

Now if i is a corner, then $\pi_i = \frac{3}{4 \cdot 3 + 24 \cdot 5 + 36 \cdot 8} = \frac{1}{140}$. Accounting for all four corners, we see that the king spends $\frac{1}{35}$ of its time in a corner.

Example: Given a room of dimensions $a \times b \times c$, repeatedly place and remove $1 \times 1 \times 1$ boxes in the following manner:

Consider the set of $1 \times 1 \times 1$ spaces in the room, which may or may not contains boxes. Of these spaces, consider those that have either a box or wall adjacent to them in the negative x, y, and z directions, and do not have one adjacent to them in the positive directions. Notice that these spaces are exactly the spaces in a perfect matching on the hex graph that can be changed.

Sample a random space from this smaller collection. If it contains a box, remove it. If it does not contain a box, then add one with some probability q. The resulting distribution is known as the **Gibbs distribution** on the boxes.

For example, the green box can be considered for removal in the following diagram, but the orange one cannot.