Week 9, Lecture 1

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1 Ising model

1.1 Definitions:

Let $G(V, E)$ be a graph. A state σ is an assignment of ± 1 to each $v \in V$ i.e. $\sigma : V \to \{+1, -1\}$. ± 1 are also referred to as spins.

Fix constants K, h, where K is the interacting strength and h, the external magnetic field strength. The energy of a configuration is defined as:

$$E(\sigma) = K \sum_{i \sim j} \sigma_i \sigma_j + h \sum_i \sigma_i$$

Ising model is a probability measure with the aforementioned energy. The probability measure is also referred to as Boltzmann or Gibbs measure given by:

$$P(\sigma) \propto \exp\left(\frac{-E(\sigma)}{T}\right)$$

where the constant of proportionality is $\frac{1}{Z}$ and $Z = \sum_\sigma \exp\left(\frac{-E(\sigma)}{T}\right)$, Z is called the partition function. If we can find Z analytically, the model is called a solvable model.

1.2 Application:

Consider a $n-$cycle i.e. $V = \{1, 2, \cdots, n\}$. For this graph, we will compute Z and correlation functions.

1.2.1 Computation of Z:

For two configurations σ and $\sigma^{'}$, consider

$$V(\sigma, \sigma^{'}) = \exp\left[\frac{-K}{T}\sigma\sigma^{'} + \frac{-h(\sigma + \sigma^{'})}{2T}\right]$$

and define a 2×2 matrix V consisting of values corresponding to different $\sigma, \sigma^{'}$ i.e.:

$$V = \begin{bmatrix} V(+,+) & V(+,-) \\ V(-,+) & V(-,-) \end{bmatrix} = \begin{bmatrix} \exp(\frac{-K}{T} + h) & \exp(\frac{K}{T}) \\ \exp(\frac{K}{T}) & \exp(\frac{-K}{T} - h) \end{bmatrix}$$

V is called a transfer matrix and the partition function $Z = tr(V^n)$, where tr refers to trace of a matrix.

Choose ϕ such that $\cot 2\phi = e^{2K} \sinh(h)$. Then the eigenvectors of V are:

$$\begin{bmatrix} \cos\phi \\ \sin\phi \end{bmatrix}, \begin{bmatrix} -\sin\phi \\ \cos\phi \end{bmatrix}$$

Let λ_1, λ_2 be the eigenvalues of V s.t. $\lambda_1 > \lambda_2$, then:

$$Z = tr(V^n) = \lambda_1^n + \lambda_2^n = \lambda_1^n \left[1 + \frac{\lambda_2^n}{\lambda_1^n}\right] \approx \lambda_1^n$$

1.2.2 Computation of correlation functions:

$$E(\sigma_1) = \frac{1}{Z} \text{ and } E(\sigma_k) = tr(SV^n) \text{ ,where } S = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

Similarly,

$$E(\sigma_1, \sigma_3) = \frac{tr(SVVSV^{n-2})}{Z}$$

i.e. S are inserted in the first and third positions in the aforementioned product of matrices.

Let

$$P = \begin{bmatrix} \cos\phi & -\sin\phi \\ \sin\phi & \cos\phi \end{bmatrix}$$

Then:

$$P^{-1}VP = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, P^{-1}SP = \begin{bmatrix} \cos 2\phi & -\sin 2\phi \\ \sin 2\phi & -\cos\phi \end{bmatrix}$$

Taking trace, we get:

$$E(\sigma_i) = \cos 2\phi, E(_i, \sigma_j) = \cos^2 2\phi + \sin^2 2\phi \left(\frac{\lambda_2}{\lambda_1}\right)^{j-i}$$

So:

$$Cov(\sigma_i, \sigma_j) = E(\sigma_i \sigma_j) - E(\sigma_i)E(\sigma_j) = \sin^2 2\phi \left(\frac{\lambda_2}{\lambda_1}\right)^{j-i}$$

which decays exponentially in $j - i$.