Week 9, Lecture 2

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1 2D Ising model

1.1 Summary

This is a $2D$ Ising model on a square lattice with periodic boundary conditions. In a $2D$ ising model, phase transition happens which doesn't in a $1D$ Ising model.

In particular, the phase transition happens at $K = \frac{\text{arcsinh}(1)}{2}$

1.2 Computation of Z:

The main idea is to use duality (hidden symmetry in the model).

Recall,

$$\sinh(x) = \frac{e^x - e^{-x}}{2}, \cosh(x) = \frac{e^x + e^{-x}}{2}$$

Then:

$$Z = \sum_{\sigma} \exp(\frac{K}{T} \sum_{i \sim j} \sigma_i \sigma_j) = \sum_{\sigma} \prod_{i \sim j} \exp(\frac{K}{T} \sigma_{ij})$$

If $\sigma = \pm 1$, then:

$$\begin{aligned} exp\left(\frac{K}{T}\sigma\right) = \cosh\left(\frac{K}{T}\right) + \sinh\left(\frac{K\sigma}{T}\right) &= \sum_{\sigma} \prod_{i \sim j} \left(\cosh\frac{K}{T} + \sinh\frac{K\sigma_i\sigma_j}{T}\right) \\ &= \sum_{\sigma} \left(\cosh\frac{K}{T}\right)^{|E|} \times \prod_{i \sim j} \left(1 + \frac{\sinh\frac{K\sigma_i\sigma_j}{T}}{\cosh\frac{K\sigma_i\sigma_j}{T}}\right) \\ &= \left(\cosh\frac{K}{T}\right)^{|E|} \sum_{\sigma} \prod_{i \sim j} \left(1 + \tanh\frac{K\sigma_i\sigma_j}{T}\right) \end{aligned}$$

When T is large, K is small $\implies K\sigma_i\sigma_j$ is small.

Let $\tanh K = e^{-K^*}$. Get partition function for Ising model with K^* and not K. $K = K^* = K_c$ when $\sinh(2K) = 1$.