MATH 607 (Applied Math II) Lecture Notes

Christophe Dethier

January 13, 2019

Lab is in 13 Pacific, from 1:00 - 2:30pm on Tuesdays.

A few last details from the raindrop problem from last week. If $N \sim \text{Pois}(\lambda)$, then $\mathbb{P}\{N = 0\} = e^{-\lambda}$. One can visualize the number of drops which intersect a point p as the volume of the cone above p in xyr space. Let's move on to some new examples.

Example: Let $N \sim \text{Pois}(\lambda)$, $X \sim \text{Binomial}(N, p)$, and $Y = N - X$. Then we claim that X and Y are independent, $X \sim \text{Pois}(\lambda)$, and $Y \sim \text{Pois}((1-p)\lambda)$. (Note that X and Y are no longer independent if one conditions on N.) To prove this claim, we will compute the joint probability generating function. If we can show that

$$\mathbb{E}\left[a^X b^Y\right] = e^{\lambda p(a-1)} e^{\lambda(1-p)(b-1)},$$

then the claim will be demonstrated. To do this, we use three different properties:

(1) $\mathbb{E}\left[u^N\right] = \sum_{n \geq 0} \frac{\lambda^n}{n!} e^{-\lambda} u^n = e^{\lambda(u-1)}$

(2) $\mathbb{E}\left[u^X | N\right] = \sum_{k=1}^{N} \binom{N}{k} (1-p)^{N-k} u^k = (1 - p + pu)^N$.

(3) $\mathbb{E}\left[f(X)\right] = \mathbb{E}\left[\mathbb{E}\left[f(X)|Y\right]\right]$. This is known as the "tower property".

First, we use (3) to condition on N;

$$\mathbb{E}\left[a^X b^Y\right] = \mathbb{E}\left[a^X b^{N-X}\right] = \mathbb{E}\left[b^N \mathbb{E}\left[\left(\frac{a}{b}\right)^X |N\right]\right].$$

Next we use (2) to evaluate the inner expectation and (1) to evaluate the outer expectation;

$$\begin{aligned}
\mathbb{E}\left[b^N \mathbb{E}\left[\left(\frac{a}{b}\right)^X |N\right]\right] &= \mathbb{E}\left[b^N \left(1 - p + p\frac{a}{b}\right)^N\right] \\
&= \exp\left(\lambda\left(b\left(1 - p + p\frac{a}{b}\right) - 1\right)\right) \\
&= \exp(\lambda((1-p)(b-1) + p(a-1))).
\end{aligned}$$

So the claim is proved.

underline: Let $\{x_1, \ldots, x_m\} \subseteq T$, where $T = [0,1]^2$ is the torus. Are they from a PPP with constant intensity? (Constant intensity means mean measure proportional to Lebesgue measure.)

To analyze this problem, we first need a general fact about PPP's, conditional uniformity: If $N \sim \text{PPP}(\mu)$ on X and $\{y_1, \ldots, y_m\}$ are points of N in $A \subseteq X$, then $\{y_1, \ldots, y_m\}$ are independent, and distributed according to Y given $N(A) = m$, where

$$\mathbb{P}\{Y \in dy\} = \frac{1}{\mu(A)} \mu(dy).$$

The idea here is to quantify "clusteriness", or over/under dispersion. To do this, let $\rho : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ be a decreasing function with $\lim_{r \to \infty} \rho(r) = 0$ and $\int_T \rho(|x|) dx = 1$. Let

$$P_i = \sum_{j \neq i} \rho(|x_j - x_i|), \qquad \text{and} \qquad P = \sum_i P_i.$$

This P measures "clusteriness" at scale ρ. Note that

$$P = \int_S \rho(|x - y|) N(dx) N(dy),$$

where $S = \{(x,y) \in T^2 : x \neq y\}$. Suppose that the points are from a PPP($\lambda dx dy$). Let's compute $\mathbb{E}[P]$.

$$\mathbb{E}[P] = \int_S \rho(|x - y|) \mathbb{E}[N(dx) N(dy)],$$

where $\mathbb{E}[N(dx)N(dy)]$ is the expected pairs of points at x and at y. That is,

$$\frac{\mathbb{E}[N(dx)N(dy)]}{dxdy} = \lim_{\epsilon rightarrow 0} \frac{1}{|B_\epsilon(x)|^2} \mathbb{E}[N(B_\epsilon(x)) N(B_\epsilon(y))] = \lambda^2$$

if $x \neq y$. So

$$\mathbb{E}[P] = \int_S \rho(|x - y|) \lambda^2 dx dy = \lambda^2.$$