MATH 607 (Applied Math II) Lecture Notes

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Last time we let $N = \sum_i \delta_{x_i}$ for some points x_i in $T = [0,1]^2$. We let ρ be a function with

$$\rho(r) = \int_T \rho(|x|)dx = 1,$$

and intuitively, we let σ be the "width", or standard deviation of ρ. Note that we could define

$$\sigma^2 = \int_T |x|^2 \rho(|x|)dx,$$

to make σ the standard deviation. Let

$$P_i = \sum_{j \neq i} \rho(|x_i - x_j|) \qquad \text{and} \qquad P = \sum_i P_i = \sum_{\substack{(i,j) \\ i \neq j}} \rho(|x_i - x_j|).$$

If N happens to be a Poisson Point Process (PPP), then $\mathbb{E}[P] = \lambda^2$, which is due to the double counting. We did this by writing this as an integral over a point measure,

$$P = \int_A \rho(|x - y|)N(dx)N(dy),$$

where $\mathbb{E}[N(dx)N(dy)] = \lambda^2 dxdy$, and $A = \{(x,y) \in T^2 : x \neq y\} \subseteq T \times T$.

Now that we've computed the mean when N is a PPP, let's also compute the variance. (To use this as a Poisson process test, we need to know the standard deviation we're expecting, as this will tell us what deviance form the mean is "reasonable".) Using standard properties of the covariance, we see that

$$\begin{aligned}
\text{var}[P] &= \text{cov}[P,P] \\
&= \text{cov}\left[\int_A \rho(|x-y|)N(dx)N(dy), \int_A \rho(|u-v|)N(du)N(dv)\right] \\
&= \int \int_A \rho(|x-y|)\rho(|u-v|)\text{cov}[N(dx)N(dy), N(du)d(dv)].
\end{aligned}$$

To analyze this last term, we consider various possibilities for x, y, u, v. If all are distinct, then $\mathrm{cov}[N(dx)N(dy), N(du)N(dv)] = 0$, because

$$\lim_{\epsilon \to 0} \frac{1}{|B_\epsilon|^4} \mathrm{cov}[N(B_\epsilon(x))N(B_\epsilon(y)), N(B_\epsilon(u))N(B_\epsilon(v))] = 0.$$

Next consider the case when $x = u$ and $y \neq v$. Then if ϵ is small enough, then

$$\mathrm{cov}[N(dx)N(dy), N(du)N(dv)] = \mathbb{E}[B_\epsilon(y))]\mathrm{var}[N(B_\epsilon(x))] = \lambda^3 |B_\epsilon|^3.$$

using the conditional covariance formula. So we arrive at

$$\mathrm{cov}[N(dx)N(dy), N(du)N(dv)] = \lambda^3 dxdydv\delta_x(u).$$

Similarly, if $x = u$ and $y = v$, then we have

$$\mathrm{cov}[N(dx)N(dy), N(du)N(dv)] = \lambda^2 dxdy\delta_x(u)\delta_y(v).$$

So overall we see that

$$\mathrm{var}(P) = 4 \int \int \rho(|x - y|)\rho(|u - v|)\lambda^3 dxdydv + 2 \int \int \rho(|x - y|)^2 \lambda^2 dxdy.$$