# MATH 607 (Applied Math II) Lecture Notes

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Today we will talk about "The" Poisson Process. Let $N \sim \text{PPP}(\lambda dx)$ on $\mathbb{R}_{\geq 0}$, and $X(t) = N([0,t])$. Let $T^k = \inf\{t \geq 0 : X_{T^{k-1}+t} > k-1\}$. That is, $T^k$ is the distance between the $k-1$th point and the $k$th point. Then $T^1, T^2, \ldots$ are iid exponential random variables with rate parameter $\lambda$. The proof of this uses the independence of PPP's on disjoint sets, and

$$\mathbb{P}\{T^1 > t\} = \mathbb{P}\{N([0,t]) = 0\} = e^{-\lambda t}.$$

Note: Assuming waiting times are exponentially distributed implies memorylessness.

Let $\phi : \mathbb{N}_{\geq 0} \to \mathbb{R}$. Then

$$\mathbb{E}[\phi(X_t)] = \sum_{n=0}^{\infty} \phi(n) e^{-\lambda t} \frac{(\lambda t)^n}{n!}.$$

Here's another method of computing this: Let $f(t) = \mathbb{E}[\phi(X_t)]$. Then

$$\frac{d}{dt} \lim_{\epsilon \to 0} \frac{\mathbb{E}[\phi(X_{t+\epsilon})] - \mathbb{E}[\phi(X_t)]}{\epsilon} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \mathbb{E}[\phi(X_{t+\epsilon}) - \phi(X_t)] = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \mathbb{E}[\phi(X_{t+\epsilon}) - \phi(X_t) | X_t].$$

Well, $X_{t+\epsilon} - X_t \sim \text{Pois}(\lambda\epsilon)$. Furthermore, for small $\epsilon$, we have $X_{t+\epsilon} - X_t = 0$ with probability roughly $1 - \lambda\epsilon$, 1 with probability roughly $\lambda\epsilon$, and $> 1$ with small probability. Therefore,

$$\mathbb{E}[\phi(X_{t+\epsilon}) - \phi(X_t) | X_t = x] = \lambda\epsilon(\phi(x+1) - \phi(x)) + \mathcal{O}(\epsilon^2).$$

Putting this together, we see that

$$\lim_{\epsilon \to 0} \mathbb{E}[\mathbb{E}[\phi(X_{t+\epsilon}) - \phi(X_t) | X_t]] = \lambda(\mathbb{E}[\phi(X_t + 1)] - \mathbb{E}[\phi(X_t)]).$$

So in conclusion,

$$\frac{d}{dt} \mathbb{E}[\phi(X_t)] = \lambda(\mathbb{E}[\phi(X_t + 1) - \mathbb{E}[\phi(X_t)]]).$$

So it follows that

$$\sum_n \phi(n) e^{-\lambda t} \frac{(\lambda t)^n}{n!}$$

is a solution to the initial value problem

$$\frac{d}{dt}\mathbb{E}[\phi(X_t)] = \lambda(\mathbb{E}[\phi(X_t+1)] - \mathbb{E}[\phi(X_t)]),$$

with the initial condition $\mathbb{E}[\phi(X_0)] = \phi(0)$. Let's verify that this power series is indeed a solution to this differential equation.

$$\begin{aligned}
\frac{d}{dt}\sum_t \phi(n)e^{-\lambda t}\frac{(\lambda t)^n}{n!} &= \sum_n \phi(n)\lambda\{n-\lambda t\}e^{-\lambda t}\frac{(\lambda t)^n}{n!} \\
&= \lambda\sum_n \{\phi(n+1)-\phi(n)\}e^{-\lambda t}\frac{(\lambda t)^n}{n!} \\
&= \lambda\sum_n \phi(n)e^{-\lambda t}\frac{(\lambda t)^{n-1}}{(n-1)!} - \lambda\sum_n \phi(n)e^{-\lambda t}\frac{(\lambda t)^n}{n!} \\
&= \sum_n \phi(n)(n-\lambda t)\frac{(\lambda t)^{n-1}}{n!}.
\end{aligned}$$

This problem should have a similar feel to the proof of Stein's Lemma on the homework. Later we will use these initial value problems as a method to solve problems without knowing the solution beforehand.

More generally, what can we say about

$$\Phi = \int \phi(x)N(dx) = \sum_i \phi(x_i),$$

where $N = \sum \delta_{x_i}$ is PPP$(\mu)$ on $S$?

**Theorem:**

$$\mathbb{E}\left[e^{iu\Phi}\right] = \exp\left\{\int_S \left(e^{iu\phi(x)}-1\right)\mu(dx)\right\}.$$

**Corollary:**

$$\mathbb{E}[\Phi] = -i\partial_u \mathbb{E}\left[e^{iu\Phi}\right]_{u=0} = \int_S \phi(x)\mu(dx).$$