607 Applied Notes

2/3/2020

Erratum for homework 4, question 2: The probability that the tower does not fall over during a given t minutes is e^{-nt}.

List of topics for Midterm 1:

1. Poisson point processes:
   - definition, assumptions
   - thinning, labeling, additivity
   - how to simulate one
   - exponential waiting times for $PP(\lambda \mathrm{d}t)$
   - characteristic function
2. General:
   - covariance
   - independence
   - conditional expectation
   - computing expectation (by conditioning)
   - law of total (co)variance
3. Continuous-time Markov chains:
   - definition(s):
     - jump rates $\to G$
     - exponential waiting times + jump probabilities
   - transition probabilities $\left(e^{tG}\right)_{xy}$
   - stationary distribution $\pi G = 0$
   - hitting probabilities: solve $Gh = 0$
   - hitting times: solve $Gh = -1$

Midterm will be 3 problems, solve 2.

Recall Q# 3) How long does an outbreak last? (on average)
Let $\tau = \inf\{t \geq 0 : X(t) = 0\}$. Find

$$\mathbb{E}[\tau | X(0) = 1]$$

Recall

$$G = \begin{bmatrix} -\delta & \delta & 0 & 0 \\ \mu & -(\mu+\lambda) & \lambda & 0 \\ 0 & 2\mu & -2(\mu+\lambda) & 2\lambda \\ 0 & 0 & 3\mu & -3(\mu+\lambda) \end{bmatrix}$$

$$G_1 = \delta$$
$$G_{n,n+1} = n\lambda$$
$$G_{n,n-1} = n\mu$$
$$G_{nn} = -n(\mu+\lambda)$$

Theorem. *Let X be a CTMC on S with generator matrix G, and $A \subset S$. Define*

$$\mathfrak{h}_A(x) = \mathbb{E}[\tau_A | X(0) = x] := \mathbb{E}^x[\tau_A]$$

where

$$\tau_a = \inf\{t \geq 0 : X(t) \in A\}$$

Then $\mathfrak{h}_A$ is the unique solution to

$$\mathfrak{h}_A(x) = 0 \quad \text{if } x \in A$$
$$G\mathfrak{h}_A(x) = -1 \quad \text{if } x \notin A$$

Corollary 0.1. *If it makes sense,*

$$\mathfrak{h}_A = (G_{-A,-A})^{-1}\,\mathbb{ONE}$$

where

$$(G_{-A,-A})_{xy} = G_{xy} \quad \text{for } x \notin A, y \notin A$$

Let $\mathfrak{h}_0(x) = \mathbb{E}^x[\tau]$. Then $\mathfrak{h}(0) = 0$. We want is $\mathfrak{h}(1) =$?

$$\delta(\mathfrak{h}(1) - \mathfrak{h}(0)) = -1$$

$$n\lambda(\mathfrak{h}(n+1) - \mathfrak{h}(n)) + n\mu(\mathfrak{h}(n) - \mathfrak{h}(n-1)) = -1$$

so

$$\mathfrak{h}(n+1) = \frac{\mu+\lambda}{\lambda}\mathfrak{h}(n) - \frac{\mu}{\lambda}\mathfrak{h}(n-1) - 1$$

How to solve this? Generating functions would work. Or could just plug it in (just need first two values, then can go from there).

$$a(n) = (\mathfrak{h}(n+1) - \mathfrak{h}(n))\left(\frac{\lambda}{\mu}\right)^n \implies a(n) - a(n-1) = -\frac{1}{n\lambda}\left(\frac{\lambda}{\mu}\right)^n$$

where we can take $a(n)$ to be a discrete analogue of the derivative of $\mathfrak{h}$ (from which we could "integrate" to obtain $\mathfrak{h}(n)$).

Proof.

$$\mathfrak{h}_A(x) = \frac{1}{-G_{xx}} + \sum_{y \neq x} \frac{G_{xy}}{-G_{xx}}\mathfrak{h}_A(y)$$

where the first term is the mean time until leaves x, the fraction on the right is the probability that it jumps from $x \to y$ next, and the rightmost term is the mean remaining time. □

Proof. We could write down proof number 2 but we're out of time. □