Applied Notes

February 12–14, 2020

Comment: Let $S(t)$ be the CMTP on $\mathbb{Z}$ that jumps up and down by one, both at rate $\frac{1}{2}$. What does S look like in the long term? It always has mean 0, and so the variance is just $\mathbb{E}[S(t)^2] = \mathbb{E}[(U(t) - D(t))^2]$, where $U(t)$ is the number of jumps up before time t and $D(t)$ is the number of jumps down. Since both $U(t)$ and $S(t)$ are Poisson with rate $\frac{t}{2}$, the variance of $S(t)$ is just t. Thus the standard deviation is $\sqrt{t}$. Fixing N, let $B_N(t) = \frac{1}{\sqrt{N}} S(Nt)$. Then $\mathbb{E}[B_N(t)] = 0$ and $\operatorname{var}(B_N(t)) = t$. This is called Brownian Scaling. The question is, does $B_N(t)$ converge in distribution to some $B(t)$ as $N \longrightarrow \infty$?

Lemma 0.0.1: If X_N and X are Markov processes such that for all s with $s < t$, all x_0, and all sets A,

$$\mathbb{P}(X_N(t) \in A \mid X_N(s) = x_0) \longrightarrow \mathbb{P}(X(t) \in A \mid X(s) = x_0),$$

then $X_N \longrightarrow X$ in distribution.

Definition 0.1: The B_N converge to some B by the lemma, and it is called **Brownian Motion**.

Theorem 0.2: If $P(t,x) = \mathbb{P}(B(t) = x \mid B(0) = 0)$, then $\frac{d}{dt}[P(t,x)] = \frac{1}{2}\frac{d^2}{dx^2}[P(t,x)]$, $P(0,x) = \delta_0(x)$, and $P(t,x) = \dfrac{1}{\sqrt{2\pi t}} e^{-x^2/2t}$ — that is, $B(t) \sim N(0,t)$.

Lemma 0.2.1: If $W \sim N(0,t)$, then $\mathbb{E}[e^{-uW}] = e^{-u^2 t/2}$. (This is the Laplace Transform, which uniquely determines probability distributions.)

Proof:

$$\mathbb{E}[e^{-uW}] = \int_{-\infty}^{\infty} e^{-ux} \cdot \frac{1}{\sqrt{2\pi t}} e^{-x^2/2t}\, dx = e^{-u^2 t/2} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi t}} e^{-(x+ut)^2/2t} dx$$

by completing the square. But that integral is just a shifted normal distribution, so it evaluates to 1. Thus $\mathbb{E}[e^{-uW}] = e^{-u^2 t/2}$.

Comment: We can also construct B another way. Let Λ be a PPP on $[0,\infty)\times\mathbb{R}$, where $[0,\infty)$ represents time and $\mathbb{R}$ represents jump size, with mean intensity $dt\cdot f(x)\cdot dx$, where f is a probability density (that is, $\int_{-\infty}^{\infty} f(x)\ dx = 1$). Now if X has probability density f, then $\mathbb{E}[X] = 0 = \int_{-\infty}^{\infty} xf(x)\ dx$ and $\mathbb{E}[X^2] = \sigma^2 = \int_{-\infty}^{\infty} x^2 f(x)\ dx$. Let $S(t)$ be the sum of the jumps out to time t. Then

$$S(t) = \sum_{\substack{(t_i,x_i)\in\Lambda \\ t_i\le t}} x_i = \int_0^t \int_{-\infty}^{\infty} x\Lambda(dt,dx).$$

Now let $B_N(t) = \frac{1}{\sqrt{N}} S(Nt) = \frac{1}{\sqrt{N}} \int_0^t \int_{-\infty}^{\infty} x\Lambda(dt,dx)$. Then

$$\begin{aligned}
\mathbb{E}[e^{-uB_N(t)}] &= \exp\left(\int_0^{tN}\int_{-\infty}^{\infty}(e^{\frac{ux}{\sqrt{N}}}-1)f(x)\ ds\ dx\right) \\
&= \exp\left(\int_0^{tN}\mathbb{E}\left[e^{\frac{uX}{\sqrt{N}}}-1\right]\ ds\right) \\
&= \exp\left(\int_0^{tN}\mathbb{E}\left[\frac{uX}{\sqrt{N}}+\frac{u^2X^2}{2\sqrt{N}}+O\left(\frac{1}{N^{3/2}}\right)\right]\ ds\right) \\
&= \exp\left(\int_0^{tN} 0+\frac{u^2\sigma^2}{2\sqrt{N}}+O\left(\frac{1}{N^{3/2}}\right)\ ds\right) \\
&= e^{-\frac{1}{2}\sigma^2u^2t}+O\left(\frac{1}{\sqrt{N}}\right)\sim N(0,\sigma^2 t).
\end{aligned}$$

Thus $B_N \longrightarrow B(\sigma^2,t)$ in distribution, since B_N is a Markov process and has iid increments — in particular, $B_N(t)-B_N(s) = B_N(t-s)$ in distribution.

Theorem 0.3: 1. B is a standard Brownian motion if and only if B is a Markov process and $B(t)-B(s)\sim N(0,t-s)$ for all $0\le s<t$.

2. The map $t\longmapsto B(t)$ is continuous but nowhere differentiable almost surely.

3. If $W(t) = \dfrac{1}{\sqrt{c}}B(ct)$ for some c, then $W(t)$ and $B(t)$ are equal in distribution.