Week 8, Lecture 1

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1 Stochastic integral

We know that

$$\int_0^t B_s ds = \int_0^t (t-s) dB_s \sim N\left(0, \frac{t^2}{2}\right)$$

But what about $\int_0^t B_s dB_s$? Is it equal to $B_t^2/2$ by normal calculus? The answer is no because it depends on how you define a stochastic integral.

Let $X_t = f(B_t)$ for some $f : R \to R$. What is dX_t? i.e. $\int_0^t dX_s := X_t - X_0$. Is it equal to $\int_0^t f'(B_s) dB_s$?

Well, take $B_0 = 0, f(0) = 0, f \in C^2$

$$f(B_t) = f(0) + B_t f'(0) + \frac{1}{2} B_t^2 f''(0) + \Theta(|B_t|^3) \tag{1}$$

$$\implies E[f(B_t)] = E[B_t f'(0)] + \frac{1}{2} t f''(0) + \Theta(t^{3/2}) \tag{2}$$

$$\implies Var[f(B_t)] = Var[B_t f'(0)] + Cov[B_t f'(0), \frac{1}{2} B_t^2 f''(0)] + Var[\frac{1}{2} B_t^2 f''(0)] + \Theta(|B_t|^3) \tag{3}$$

$$= [f'(0)]^2 t + \Theta(|t|^2) \tag{4}$$

2 Simulation

We can simulate X_t as follows:

$$
\begin{aligned}
& t = 0, B_0 = 0 \\
& X_0 = f(0) \\
& \text{while } \ t < T: \\
& \quad W \sim N(0,1) \\
& \quad B(t+dt) += \sqrt{dt}W \\
& \quad X(t+dt) += \frac{1}{2} f''(B_t)dt + |f'(B_t)|\sqrt{dt}W \\
& \quad t += dt
\end{aligned}
$$

3 Ito's Lemma

$$df(B_t) = f'(B_t)dB_t + \frac{1}{2} f''(B_t)dt \tag{5}$$

$$\text{i.e. } f(B_t) = f(B_0) + \int_0^t f'(B_s)dB_s + \frac{1}{2}\int_0^t f''(B_s)ds \tag{6}$$

where $\int f'(B_s)dB_s$ is the Ito integral.
Ito integral is defined as:

$$\lim_{N\to\infty} \sum_{k=0}^{N-1} g\left(B\left(\frac{kt}{N}\right)\right)\left[B\left(\frac{(k+1)t}{N}\right) - B\left(\frac{kt}{N}\right)\right] =: \int_0^t g(B_s)dB_s$$