Week 8, Lecture 2

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1 Ito's lemma

If $X_t = f(B_t)$, then

$$dX_t = f'(B_t)dB_t + \frac{1}{2}f''(B_t)dt$$

2 Ito's integral

If $G(x) = \int_0^x g(z)dz$, then

$$\int_0^t g(B_s)dB_s = G(B_t) - \frac{1}{2}\int_0^t g''(B_s)ds \tag{1}$$

$$= \lim_{N\to\infty} \sum_{k=0}^{N-1} g\left(B\left(\frac{kt}{N}\right)\right)\left[B\left(\frac{(k+1)t}{N}\right) - B\left(\frac{kt}{N}\right)\right] \tag{2}$$

3 Stieltjes integral

If $F : [0, t] \to R, g : R \to R$, then

$$\int_0^t g(s)dF_s = \lim_{N\to\infty} \sum_{k=0}^{N-1} g\left(g\left(\frac{kt}{N}\right)\right)\left[F\left(\frac{(k+1)t}{N}\right) - F\left(\frac{kt}{N}\right)\right] \tag{3}$$

$$= \lim_{N\to\infty} \sum_{k=0}^{N} g\left(g\left(\frac{kt}{N}\right)\right)\left[F\left(\frac{kt}{N}\right) - F\left(\frac{(k-1)t}{N}\right)\right] \tag{4}$$

(3) and (4) are equal if F has countably many discontinuities and has finite quadratic variation i.e.

$$\lim_{N\to\infty} \sum_{k=0}^{N-1} \left[F\left(\frac{(k+1)t}{N}\right) - F\left(\frac{kt}{N}\right)\right]^2 < \infty$$

4 Quadratic variation of B_t

Lemma: The quadratic variation of B_t is t i.e.

$$\lim_{N\to\infty}\sum_{k=0}^{N-1}\left[B\left(\frac{(k+1)t}{N}\right)-B\left(\frac{kt}{N}\right)\right]^2 = t \text{ a.s.}$$

Proof:

$$Z_k = \left[B\left(\frac{(k+1)t}{N}\right)-B\left(\frac{kt}{N}\right)\right]\sqrt{N}$$

are iid $\sim N(0,t)$ for $0 \le k \le N-1$. So:

$$\frac{1}{N}\sum_{k=0}^{N-1} Z_k^2 \xrightarrow[N\to\infty]{SLLN} EZ_1^2 = t$$

Hence proved.

5 Examples and a lemma

$f(x) = x^2 \implies X_t = B_t^2$. We know that $EX_t = t$. By Ito, $dX_t = 2dB_t + dt$ i.e. $B_t^2 = X_t = t + \int_0^t 2B_s dB_s$

Lemma: $E\left[\int_0^t f(B_s)dB_s\right] = 0$

Proof:

$$E\left[\frac{1}{N}\sum_{k=0}^{N-1} f\left(B\left(\frac{kt}{N}\right)\right)\left[B\left(\frac{(k+1)t}{N}\right)-B\left(\frac{kt}{N}\right)\right]\right] \tag{5}$$

$$= \frac{1}{N}\sum E\left[f\left(B\left(\frac{kt}{N}\right)\right)\left[B\left(\frac{(k+1)t}{N}\right)-B\left(\frac{kt}{N}\right)\right]\right] \tag{6}$$

Thus $EB_t^2 = EX_t = t + 0 = t$

By definition of Z_k,

$$B\left(\frac{(m+1)t}{N}\right) = \frac{1}{\sqrt{N}}\sum_{k=0}^{m} Z_k \tag{7}$$

$$\implies B_t^2 = \frac{1}{N}\sum_{k=0}^{N-1}\sum_{j=0}^{N-1} Z_k Z_j \tag{8}$$

$$= \frac{1}{N}\sum_{k=0}^{N-1} Z_k^2 + \frac{2}{N}\sum_{k=1}^{N-1}\sum_{j=0}^{k-1} Z_k Z_j \xrightarrow{N\to\infty} t + 2\int_0^t B_s dB_s \tag{9}$$