Week 8, Lecture 3

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1 Diffusions

Let $X_t \in R$ be the solution to the SDE

$$dX_t = a(X_t)dt + b(X_t)dB_t$$

then X_t is called a diffusion process with drift $a(x)$ and diffusion $b(x)$.

2 Simulation of a diffusion process

Started from x, until T:

$$\begin{aligned}
&\text{Let } \; X_0 = x \\
&t = 0 \\
&\text{while } \; t < T : \\
&\quad W \sim N(0, dt) \\
&\quad X(t+dt) = X(t) + dta(X_t) + Wb(X_t)
\end{aligned}$$

Equivalently, if X_t is a time-homogenous Markov process such that:

$$E[X(t+dt) - X(t)] = a(X_t)dt + \Theta(dt^2) \tag{1}$$

$$Var[X(t+dt) - X(t)] = b(X(t))^2 dt + \Theta(dt^2) \tag{2}$$

Then:

$$X_t = X_0 + \int_0^t a(X_s)ds + \int_0^t b(X_s)dB_s$$

Recall the example:

$$X_t = B_t^2 \tag{3}$$
$$dX_t = 2B_t dB_t + dt \tag{4}$$
$$B_t^2 = B_0^2 + \int_0^t 2B_s dB_s + t \tag{5}$$

3 Example

ODE: Let us say we solve $f(t) = 2\sqrt{f(t)}$, then $f(t) = f(0) + t^2$
SDE: Let us say we solve $dX_t = 2\sqrt{X_t} dB_t + dt$, then $X_t = B_t^2 + X_0$ is the unique solution.

4 Theorems

Let $p(t, x, y) = P_x\{X_t = y\}$. Then this is the unique solution to $\frac{d}{dt} p(t, x, y) = a(x)\frac{d}{dx} + \frac{1}{2} b(x)^2 \frac{d^2}{dx^2} p(t, x, y)$ with the boundary condition $p(t, x, y) \xrightarrow{t \to 0} \delta_x(y)$.

Recall: For B_t,

$$p(t, x, y) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{(x-y)^2}{2t}}$$

solves the equation

$$\frac{d}{dt} p(t, x, y) = \frac{1}{2} \frac{d^2}{dx^2} p(t, x, y)$$

Definition: The generator a CTMP $X_t \in S$ is $G : f \to Gf$ with $f : S \to R, GF : S \to R$ is defined as follows:

$$Gf(x) = \lim_{\epsilon \to 0} \frac{E_x[F(X_\epsilon) - f(x)]}{\epsilon}$$

Example: If X_t is a CTMC on $\{1, 2 \cdots n\} = S$. So if $f : S \to R$, then $f \in R^n$ and $G \in R^{n \times n}$ with

$$G_{ij} = \begin{cases} \textbf{jump rates if } i \neq j \\ \textbf{- Sum of jump rates if } i = j \end{cases}$$

Theorem: $\frac{d}{dt} p(t, x, y) = G_x p(t, x, y)$
Proof: For diffusions, let $dX = a(X)dt + b(X)dB$, $G = a(x)\frac{d}{dx} + \frac{1}{2} b(x)^2 \frac{d^2}{dx^2}$. Let $F_t(x) = E_x[f(X_t)]$. Then:

$$
\begin{aligned}
F_{t+\epsilon}(x) &= E_x[f(X_{t+\epsilon})] && (6)\\
&= E_x[E[f(X_{t+\epsilon})|X_\epsilon]] && (7)\\
&= E_x[F_t(X_\epsilon)]\textbf{By time homogeneity and Markov property} && (8)\\
& && (9)
\end{aligned}
$$

So

$$
\begin{aligned}
\frac{d}{dt}F_t(x) &= \lim_{\epsilon\to 0}\frac{F_{t+\epsilon}(x) - F(x)}{\epsilon} && (10)\\
&= \lim_{\epsilon\to 0}\frac{E_x[F_t(X_\epsilon)] - F(x)}{\epsilon} && (11)\\
&= GF(x)\ \textbf{By definition of } G && (12)
\end{aligned}
$$

So

$$
\begin{aligned}
F_t(x) &= E_x[f(X_y)] && (13)\\
&= \int_R p(t,x,y)f(y)dy && (14)\\
\implies \frac{d}{dt}F_t(x) &= \int_R \frac{d}{dt}p(t,x,y)f(y)dy && (15)\\
\implies Gf_t(x) &= \int_R \left(a(x)\frac{d}{dx} + \frac{1}{2}b(x)^2\frac{d^2}{dx^2}\right)p(t,x,y)f(y)dy && (16)
\end{aligned}
$$

This is true for all f, so $\frac{d}{dt}p(t,x,y) = G_x p(t,x,y)$